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The overall goal of this AFOSR sponsored research program is to construct fast and efficient numerical algorithms for solving stochastic partial differential equations and apply them to solve optimal control problems under uncertainty which is described by stochastic partial differential equations. It is well understood that effective numerical methods for stochastic partial differential equations (SPDES) are essential for uncertainty quantification. In the last decade much progress has been made in the construction of numerical algorithms to efficiently solve SPDES with random coefficients and white noise perturbations. However, high dimensionality and low regularity are still the bottleneck in solving real world applicable SPDES with efficient numerical methods. This project is intended to address the mathematical aspects of numerical approximations of SPDES, including error analysis and complexity analysis and development of new efficient numerical algorithms.

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Numerical Solutions for Optimal Control Problems under SPDE Constraints

FA9550-08-1-0119

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1 Introduction

The overall goal of our AFOSR sponsored research program is to construct fast and efficient numerical algorithms for solving stochastic partial differential equations and apply them to solve optimal control problems under uncertainty which is described by stochastic partial differential equations.

Uncertainty quantification has been an active research area in the past 10 years with many significant applications ranging from signal processing to aircraft wing designs. It is well understood that effective numerical methods for stochastic partial differential equations (SPDES) are essential for uncertainty quantification. In the last decade much progress has been made in the construction of numerical algorithms to efficiently solve SPDES with random coefficients and white noise perturbations. However, high dimensionality and low regularity are still the bottleneck in solving real world applicable SPDES with efficient numerical methods. This project is intended to address the mathematical aspects of numerical approximations of SPDES, including error analysis and complexity analysis and development of new efficient numerical algorithms. Our contributions toward this objective include

(i). *Optimal Kronecker sequence for the quasi Monte Carlo method and the Monte Carlo method for stochastic Stokes equations.* The generation of appropriate high-quality quasirandom sequences (low-discrepancy sequences) is crucial to the success of quasi-Monte Carlo methods. We present a new algorithm for finding an optimal Kronecker sequence within choices of irrationals. we illustrated with numerical experiments why our algorithm is efficient for breaking these correlations.

(ii). *Efficient algorithms for stochastic partial differential equations.* We are interested in the the orthogonal polynomial expansion of **d-variable** functions. Fast and efficient algorithms of evaluating such expansions are essential for efficiently solving stochastic partial differential equations with spectral methods where d may be fairly large.

(iii). *High order numerical method for nonlinear filter problems.* We first convert the nonlinear filter problem into a problem of solving a stochastic partial differential equations (SPDES). We then construct a high order method to solve the SPDES and apply our algorithm to solve a ship tracking problem.

2 Brief Overview of Accomplishments

During the prior three-year grant period we have published 12 papers that have been supported by AFOSR funding. Selections of our current accomplishments are summarized here to provide an introduction to the concepts we have developed in the past and upon which future planned studies are based.

2.1 The optimal Kronecker Sequence for quasi Monte Carlo simulations

Kronecker sequences are easy and fast to implement. However, problems with Kronecker sequences arise from correlations between the choice of parameters which are used for different dimensions. These correlations cause Kronecker sequences to have poor two-dimensional projections for some pairs of dimensions.

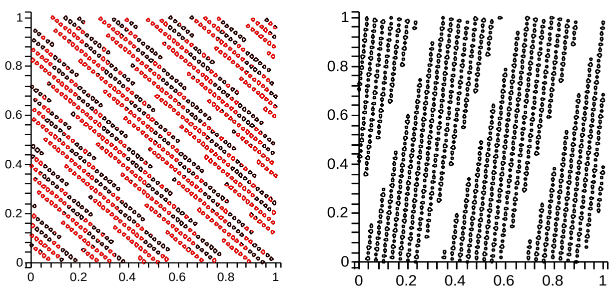

Figure 1: Left: 1000 points of dimension 1 and 2 projection for original Kronecker sequence; right: 1000 points of dimension 26 and 27 projection for original Kronecker sequence.

In Figure 1, we can see high correlations and similarities in the poor two-dimensional projections for original Kronecker sequences.

There are at least two possible ways to break the correlations we have seen in the Kronecker sequence. One is by increasing the difference between the bases for any pair of dimensions; the other is to select better basis for the Kronecker sequence. It became clear that the smaller the partial quatients in the coninued fractions of the irrarional number, the more uniformly distributed the s-dimenional Kronecker sequence is [4]. We blend those two methods and propose a novel approach by using generarlized golden ratio, which is widely applied in many applications in computer science [6]. The 2-D projections derived from our new algorithm are shown in Figure 2. The patterns appeared in Figure 1 disappear in Figure 2.

We have demonstrated that generalization of golden ratio is the simplest and most effective method to break this correlation in original Kronecker sequences. It is also very easy to implement the original Kronecker sequences and generate high-quality of the optimal Kronecker sequences.

In Bratley et al's paper [1], the Faure and Sobol sequences are used to evaluate high dimensional integrals, and the errors in the numerical results for over 30 dimensions become quite large. When we compared their results with our optimal Kronecker sequences, we found that our method has much smaller errors comparing to the

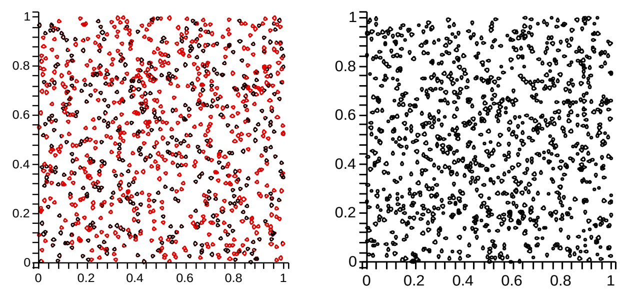

Figure 2: Left: 1000 points of dimension 1 and 2 projection for optimal Kronecker sequence; right: 1000 points of dimension 26 and 27 projection for optimal Kronecker sequence.

benchmark Faure and Sobol methods.

2.2 High order numerical method for nonlinear filter problems

Assume that $X = X_t$ is a signal process and $Y = Y_t$ is the observation. Because of noises, X and Y are solutions of the following stochastic differential equations.

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dU_t \\ dY_t &= h(X_t)dt + dV_t \end{cases}$$

Here U_t, V_t are independent Brownian motions which reflect the noises of the signal and the observation. The filter problem is to seek the best estimation of $\phi(X)$ given the observation Y. Mathematically this amounts to seek a conditional expectation $E\left(\phi(X_t) \mid (Y_s, 0 \le s \le t)\right)$, i.e.,

$$E\left(\phi(X_t) \mid (Y_s, 0 \le s \le t)\right) = \underset{\{Z_t \text{ is } Y_{(0,t)} \text{ measurable}\}}{\text{Argmin}} \left(E\left|\phi(X_t) - Z_t\right|^2\right)$$

According to Zaikai filter theory, the best approximation is given by

$$E\left(\phi(X_t) \mid (Y_s, 0 \le s \le t)\right) = \frac{\int \phi(x)p(t,x)dx}{\int p(t,x)dx}.$$

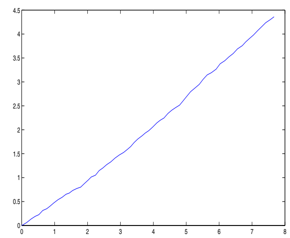

Figure 3: Sample path of the ship location

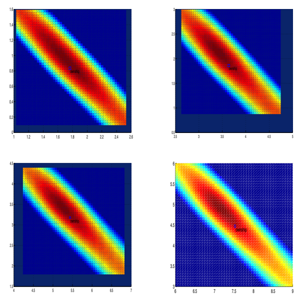

Figure 4: Un-normalized probability distribution of the ship location

Also according to the Zakai's theory, p satisfies the following stochastic partial differential equation.

$$dp_t = (\sum_{i,j=1}^{d} \sigma\sigma^* \frac{\partial^2 p_t}{\partial x_i \partial x_j} + \sum_{i=1}^{p} b_i \frac{\partial p_t}{\partial x_i})dt + p_t h^* dY_t.$$

So the nonlinear filter problem is solved as long as the above stochastic partial differential equation is be solved. We have derived a higher order method to solve the nonlinear stochastic differential equations. We achieve the high order method through solving a backward forward doubly stochastic differential equations. Detail of the method can be found in [3].

We then applied the method to solve a problem of tracking a ship in open water through the angle observation of the ship. Figure 3 shows a sample path of the ship and Figure 4 shows the probably distributions of the ship location at four different times.

2.3 Error analysis of finite element approximations for stochastic Stokes equations

In this subproject, we consider the steady-state Stokes equation with the forcing term perturbed by white noise:

$$\begin{cases} -\nu\Delta u(x,\xi) + \nabla p(x,\xi) &= f + \sigma(x)\dot{W}(x,\xi) \quad x \in \ D \subset R^d(d=2,3), \quad \xi \in \Omega, \\ \text{div} u &= 0 \quad \text{in } D, \\ u &= 0, \quad \text{on } \partial D. \end{cases} \tag{1}$$

Here $(\Omega, \mathcal{F}, P)$ is a probability space and $\dot{W} = (\dot{W}^1, \ldots, \dot{W}^d)$ is the white noise such that

$$E(\dot{W}^j(x)\dot{W}^j(x')) = \delta(x - x'), \quad x,\ x' \in \Omega, \quad j = 1, \ldots, d.$$

Our strategy of solving this problem consists of two steps. In the first step, we construct an approximate white noise $\dot{W}_h$ based the finite element partition of the physical domain D. In the second step, we construct a finite element approximation (u_h, p_h) for the solution (u, p) of (1). Under certain assumptions on the regularity of the domain and using the Taylor-Hood finite element, we proved the following error estimates [2].

$$\begin{cases} E\left(\|u - u_h\|^2 + \|p - p_h\|_{-1}^2\right) \le C|\ln h|h^2 \quad d = 2, \\ E\left(\|u - u_h\|^2 + \|p - p_h\|_{-1}^2\right) \le Ch \quad d = 3. \end{cases} \tag{2}$$

The significance of the error estimate is twofold:

- It indicates that the due to the lack of regularity, the convergence order of the finite element solution for stochastic Stokes equations is much lower than that for deterministic Stokes equations.
- It provides a guidance for the number of samples used in Monte Carlo simulations to evaluate the statistics of the solutions.

We also constructed practical numerical algorithms to find the finite element approximations using the Monte Carlo method and verified our theoretical results through numerical experiments ([2]).

2.4 Fast and efficient numerical algorithms for stochastic partial differential equations

We first developed a fast algorithm of high dimension orthogonal polynomial expansions on sparse grids. Then we apply this algorithm to solve stochastic partial differential equations (SPDEs). This is achieved through the following four steps.

- **Step 1.** Use only sparse indices in the orthogonal expansions. For a d dimensional problem, this reduces the number of terms in the orthogonal expansion from $O(n^d)$ to $O(n \log^{d-1} n)$, a tremendous saving of computing cost.

- **Step 2.** Use fast Fourier transform as well as sparse grid to compute the coefficients of the orthogonal expansion. This reduces the computing cost from $O(n^d)$ to $O(n \log^{(d+1)n})$, yet another significant reduction of computing complexity.

- **Step 3.** Apply the fast orthogonal expansion algorithm to solve stochastic partial differential equations with random coefficients and forcing term using the spectral method. The remarkable feature of our algorithm is that the coefficient matrix is sparse and each element can be calculated analytically.

Our numerical experiments show that the overall computing complexity of solving the stochastic partial differential equation is in the same magnitude as computing the orthogonal expansion of the forcing term, which is proved to be $O(n \log^{d+1} n)$. In comparison, the complexity of conventional spectral method is at least $O(n^d)$, which is prohibitively expensive when the dimensional d is large as in the case of solving stochastic partial differential equations. The numerical verification of the computational complexity is shown in Figure 2.4.

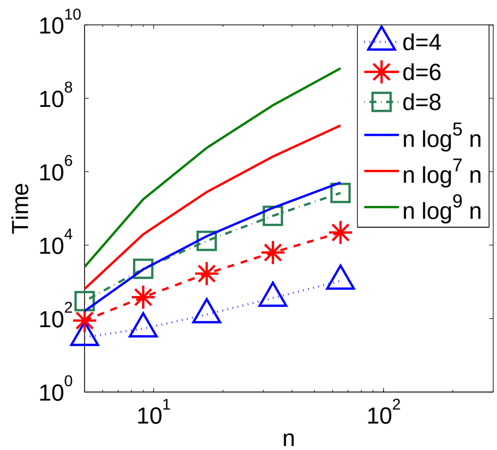

Figure 5: The CPU times in numerical experiment v.s. theoretical estimates Solid lines: O($n \log^{d+1} n$),$d = 4, 6, 8$ Dashed lines: CPU times in numerical experiment

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4 Publications

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